Lecture 4 : Product spaces and independence

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4.1 Product spaces and Fubini's Theorem

Definition 4.1 *If $(\Omega_i, \mathcal{F}_i)$ are measurable spaces, $i \in I$ (index set), form $\prod_i \Omega_i$. For simplicity, $\Omega_i = \Omega_1$.*

$\prod_i \Omega_i$ (write Ω for this) is the space of all maps: $I \to \Omega_1$. For $\omega \in \prod_i \Omega_i$, $\omega = (\omega_i : i \in I, \omega_i \in \Omega_i)$. Ω is equipped with projections, $X_i : \Omega \to \Omega_i$, $X_i(\omega) = \omega_i$.

[picture of square, Ω_1 on one side, Ω_2 on other, point ω in middle, maps under projection to each side]

Definition 4.2 *A product σ-field on Ω is that generated by the projections: $\mathcal{F} = \sigma\left((X_i \in F_i) \mid F_i \in \mathcal{F}_i\right)$.*

$F_1 \times F_2 = (\Omega \mid \Omega_1 \in F_1 \wedge \Omega_2 \in F_2) \in \mathcal{F} = (X_1 \in F_1) \cap (X_2 \in F_2)$.

We now seek to construct the *product measure*. We start with the $n = 2$ case. $(\Omega, \mathcal{F}) = (\Omega_1, \mathcal{F}_1) \times (\Omega_2, \mathcal{F}_2)$. Suppose we have probability measures P_i on $(\Omega_i, \mathcal{F}_i)$, $i = 1, 2$. Then there is a natural way to construct $P = P_1 \times P_2$ (product measure) on $(\Omega, \mathcal{F})$.

Key idea: $F_1 \in \mathcal{F}_1$, $F_2 \in \mathcal{F}_2$, $P(F_1 \times F_2) = P_1(F_1) \times P_2(F_2)$.

Theorem 4.3 (Existence of product measure and Fubini theorem) *: (notation $\omega = (\omega_1, \omega_2)$) There is a unique probability measure P on $(\Omega_1, \mathcal{F}_1) \times (\Omega_2, \mathcal{F}_2)$ such that for every non-negative product-measurable [measurable w.r.t. the product σ-field] function $f : \Omega_1 \times \Omega_2 \to [0, \infty)$,*

$$\int_{\Omega_1 \times \Omega_2} f(\omega_1 \times \omega_2) P(d\omega) = \int_{\Omega_1} \left[\int_{\Omega_2} f(\omega_1, \omega_2) P_2(d\omega_2) \right] P_1(d\omega_1) =$$
$$\int_{\Omega_2} \left[\int_{\Omega_1} f(\omega_1, \omega_2) P_1(d\omega_1) \right] P_2(d\omega_2).$$

Note that this really tells very explicitly what P is:

$$P(A) = \int_{\Omega_2} 1_A(\omega) P(d\omega) \tag{$*$}$$

Fix ω_2, look at $A\omega_2 = \{\omega_1 \mid (\omega_1, \omega_2) \in A\}$.

$$\int P_1(A\omega_2) P_2(d\omega_2)$$

Look at the level of sets. $f = 1_A$. Look at formula $(*)$. Look at the collection $\mathcal{C}$ of all $A \subset \Omega_1 \times \Omega_2$ such that $*$ makes sense, i.e.:

1. $A\omega_2 \in \mathcal{F}_1$ for every $\omega_2 \in \Omega_2$.
2. $\omega_2 \to P_1(A\omega_1)$ is measurable.

Observe:

1. $\mathcal{C}$ contains all $A_1 \times A_2$ (π-system, closed under intersection), and get $P(A_1 \times A_2) = P_1(A_1) P_2(A_2)$
2. $\mathcal{C}$ is a λ-system.

Thus, $\mathcal{C} \supset \sigma(A_1 \times A_2)$, which is the product of σ-fields.

(Checking λ-system uses monotone convergence theorem.)

We know that the order of integration was not relevant, because the other way gives us a measure which agrees on rectangles (by commutativity of addition), and rectangles generate (and we get a π-system out of them), so we can apply the π-λ-theorem.

Extend our measure on indicator functions to simple functions, additively, and so on.

4.2 Independence

Random variables X_1 and X_2 with values in $(\Omega_1, \mathcal{F}_1)$, $(\Omega_2, \mathcal{F}_2)$ are called *independent* iff $\mathbb{P}(X_1 \in F_1, X_2 \in F_2) = \mathbb{P}(X_1 \in F_1)\mathbb{P}(X_2 \in F_2)$ for all $F_1 \in \mathcal{F}_1$, $F_2 \in \mathcal{F}_2$.

Observe:

1. If we take $\Omega = \Omega_1 \times \Omega_2$ and $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ and $\mathbb{P} = P_1 \times P_2$, and $X_i(\omega) = \omega_i$ projections as before, then X_1 and X_2 are independent random variables with distributions P_1 and P_2.

2. If X_1 and X_2 are independent random variables, defined on any background space $(\Omega, \mathcal{F})$, then the *joint distribution* of (X_1, X_2) is the product measure $P_1 \times P_2$, $P_i = P_{X_i}$ which is the $\mathbb{P}$ distribution of X_i.

$\omega \to X_1(\omega)$, $\omega \to X_2(\omega)$. Look at $\omega \to (X_1(\omega), X_2(\omega))$ as a map from Ω to $\Omega_1 \times \Omega_2$.

We check that this map is product-measurable. If $A_1 \in \mathcal{F}_1$ and $A_2 \in \mathcal{F}_2$, then

$$\{(X_1, X_2) \in A_1 \times A_2\} = (X_1 \in A_1) \cap (X_2 \in A_2) \in \mathcal{F}.$$

Now $P_{(X_1,X_2)} = \mathbb{P}$ distribution of (X_1, X_2) makes sense.

Corollary 4.4 (Fubini) *If X_1 and X_2 are independent random variables,*

$$\mathbb{E}\left[f(X_1, X_2)\right] = \int \left[\int f(x_1, x_2) P_1(dx_1)\right] P_2(dx_2).$$

This justifies formulas for distribution of $X_1 + X_2$, $X_1 X_2$ for real r.v.'s. Example: If X_1 has density f_1, X_2 has density f_2, $\mathbb{P}(X_1 \in A) = \int_{A_1} f(x_1) dx_1$. Then $X_1 + X_2$ has a density

$$f(z) = (f_1 \times f_2)(z) = \int_{\mathbb{R}} f_1(x) f_2(z - x) dx.$$

Check this by application of Fubini theorem.

$\mathbb{E}[g(X_1 + X_2)] = \int g(z) f(z) dz$.

Useful fact: For real random variables, X_1 and X_2 are independent if and only if $\mathbb{P}(X_1 \leq x_1, X_2 \leq x_2) = \mathbb{P}(X_1 \leq x_1)\mathbb{P}(X_2 \leq x_2)$ for all real x_1, x_2.

Fix $F_1 = (-\infty, x_1]$ first. Consider all sets with $\mathbb{P}(X_1 \in F_1, X_2 \in F_2) = \mathbb{P}(X_1 \in F_1)\mathbb{P}(X_2 \in F_2)$.

Extend to n variables. $X_1, \ldots, X_n$ are independent iff

$$\mathbb{P}\left(\bigcap_i (X_i \in F_i)\right) = \prod_i \mathbb{P}(X_i \in F_i).$$

Same discussion with product spaces. Most proofs work by induction on n, reduces to $n = 2$. For example, X_1, X_2, X_3 are independent iff X_1 and X_2 are independent and (X_1, X_2) and X_3 are independent.

Intuitive properties: e.g. if $X_1, \ldots, X_5$ are independent, then $X_1 + X_3 + X_5$ and $X_2 + X_4$ are independent.

To check this, we need to check that we can factor the probabilities as above, which goes for simple functions, and then extends.

Proposition 4.5 *If X and Y are independent and $\mathbb{E}(|X|) < \infty$ and $\mathbb{E}(|Y|) < \infty$ then $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$.*

Proof: Use Fubini. Be careful: we only showed Fubini for non-negative functions.

First, check in case $X \geq 0$, $Y \geq 0$. $\mathbb{E}(XY) = \int_{\mathbb{R}} \int_{\mathbb{R}} xy \mathbb{P}(X \in dx)\mathbb{P}(Y \in dy) = \mathbb{E}(X)\mathbb{E}(Y)$, by Fubini.

In general, $X = X^+ - X^-$, $Y = Y^+ - Y^-$, and get four pieces, then put back together. ■

Note that the most general form of Fubini's theorem gives

$$\mathbb{E}(f(X,Y)) = \int \int f$$

provided this $\int \int f$ formula is finite when f is replaced by $|f|$.]

Recall $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$ always, provided they are finite. $\mathbf{V}ar(X+Y) = \mathbf{V}ar(X) + \mathbf{V}ar(Y) + 2\mathbb{E}\left[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))\right]$ (last term is *covariance*), and last term is 0 if X and Y are independent.